Looking into Hidden Worlds
- using Mathematics-

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Let us start with looking at the following problem. Assume that we are given the following array of the first 16 natural numbers.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

This arrangement as a 4x4 array of numbers is sometimes called a 4x4 *matrix*.

Now rearrange the numbers in the 4×4 array

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

to obtain a new 4×4 array with the following properties:

1. The sum of the numbers in each row equals 34.
2. The sum of the numbers in each column equals 34.
3. The sum of the numbers on the two main diagonals equals 34.

Here is the result to this problem. It is one of the famous magic squares

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

It has indeed quite a few remarkable properties:

1. The sum of the numbers in each row equals 34.
2. The sum of the numbers in each column equals 34.
3. The sum of the numbers on the two main diagonals equals 34.
4. The sum of the numbers in each of the four 2×2 corner quadrants equals 34.
5. The sum of the numbers in the center 2×2 quadrant equals 34.
6. The sum of the numbers in the four corner cells equals 34.

Magic Squares have become famous in Art and Literature

Figure: Melencolia I, engraving by Albrecht Dürer

Source: reference 1.

Magic Squares

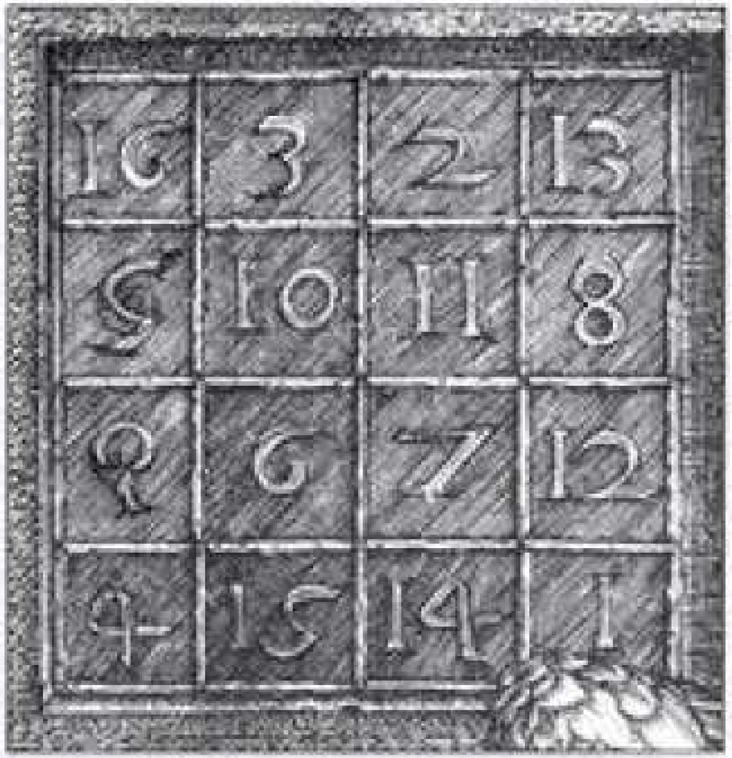


Figure: Table in the upper right corner of the engraving

Source: reference 1.

Historically, after the first magic square was found, the question arose whether it is the only one with the above properties, or whether there are other, different magic squares.

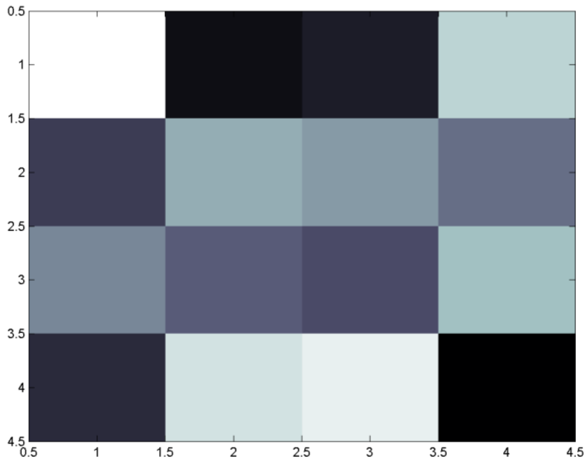
Soon others were found, like the following one:

$$\begin{bmatrix} 1 & 4 & 14 & 15 \\ 13 & 16 & 2 & 3 \\ 8 & 5 & 11 & 10 \\ 12 & 9 & 7 & 6 \end{bmatrix}$$

A systematic search for ‘magic squares’ with all kind of properties started and is still ongoing.

Much later, and independently, a similar problem arose with the development of tomography:

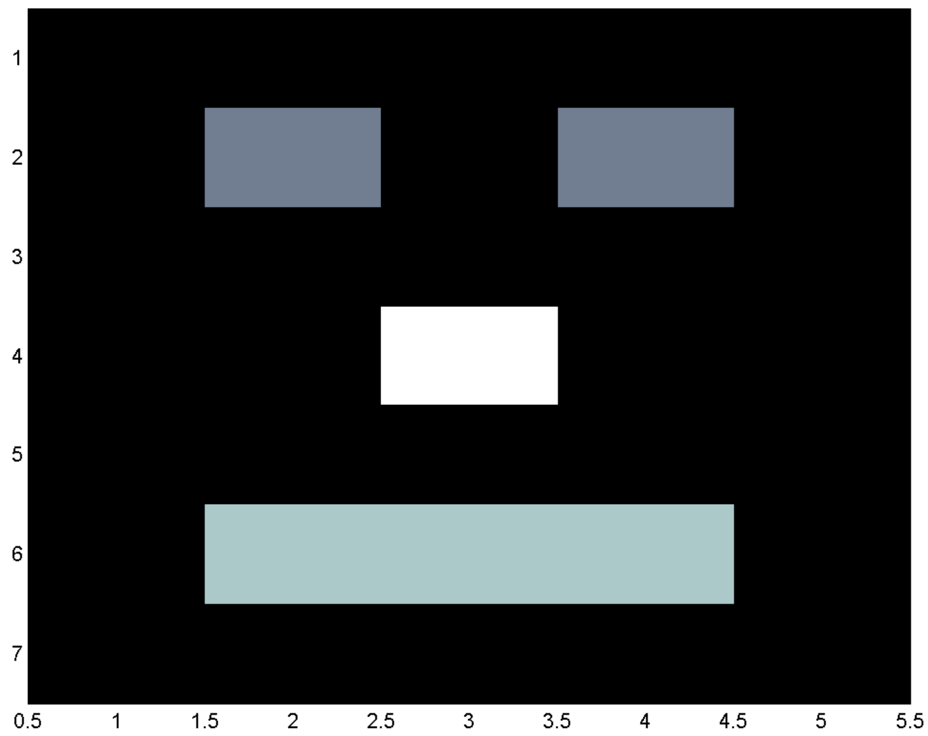
$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$



We can encode numbers by colours or grayscale values in order to visualize rectangular arrays: The higher the value, the brighter the pixel. This is similar to a black-and-white photograph of the array.

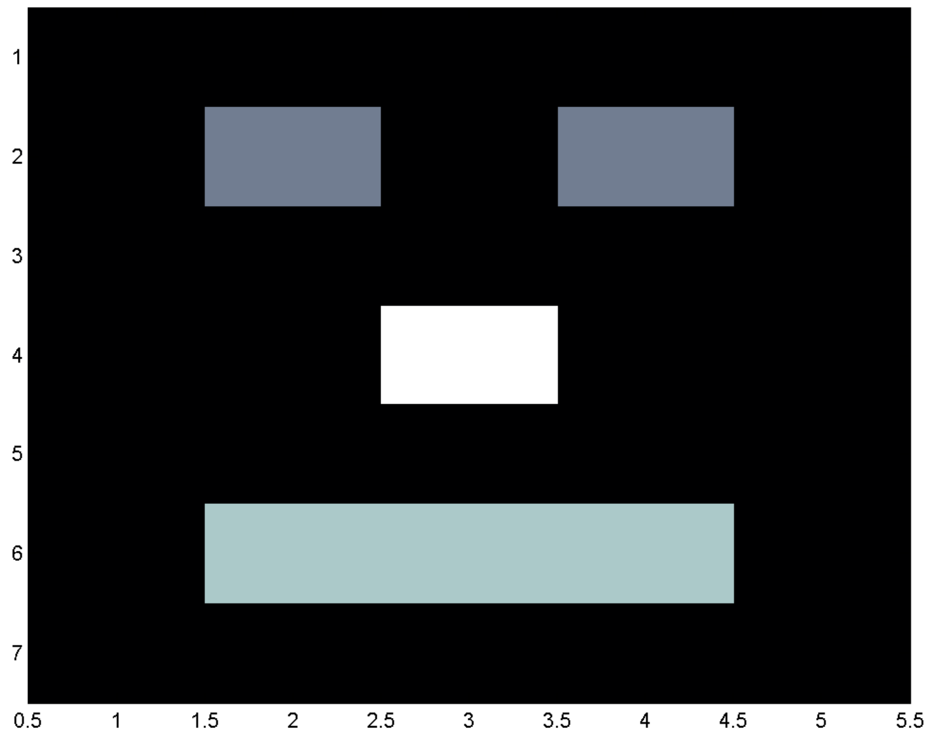
Here is a 7x5 array. Do you see how to ‘paint’ with numbers?

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$



This way it is easier to see

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$



The bigger the number the brighter the pixel!

And here is a 367x490 example which can be interpreted as an X-ray image of the human spine. The corresponding array of numbers is too large to be displayed here, but you can now imagine how it might look like….

Source: reference 1.

During a **regular X-ray screening** an X-ray beam created by a tube is sent through the body and hits a photo-sensitive screen behind the patient. Along the path the X-rays are absorbed stronger by dense tissue (e.g. bones) than by soft tissue. The recorded data represent therefore the sum of density values along the ray trajectories through the body.

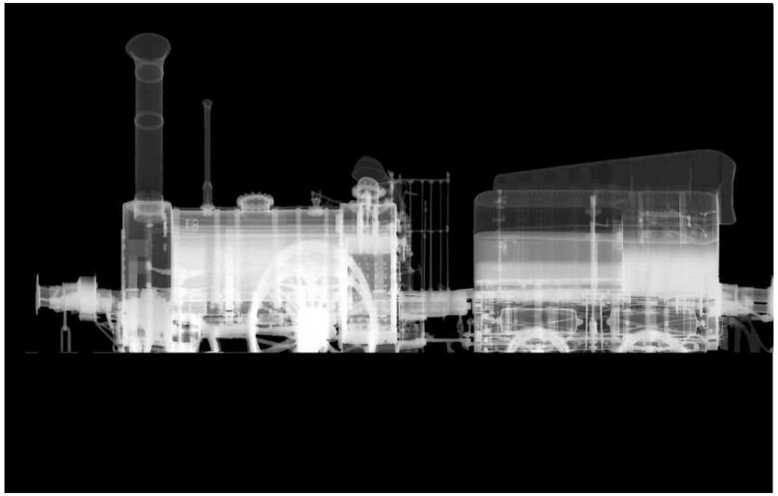
Source: reference 2.

From these sums of densities along ray paths we obtain a grayscale image of the body densities along these rays.

Source: reference 5.

Bright pixels on the X-ray image indicate an accumulation of ‘dense’ material on the ray path, such as a bones. Dark areas correspond to soft tissue or lungs filled with air.

X-rays are nowadays used to image almost everything.
Here is an X-ray of Stephenson's Planet Locomotive



Source: reference 6.

Planet was an early steam locomotive built in 1830 by Robert Stephenson for the Liverpool and Manchester Railway. A replica of the locomotive is maintained by **Manchester's Museum of Science and Industry**

A big Breakthrough in imaging came with the development of Computerized Tomography

Source: reference 2.

The **Nobel Prize in Medicine 1979** was awarded to A.M. Cormack and G.N. Hounsfield for their contributions towards the development of Computerized Tomography

Source: reference 12.

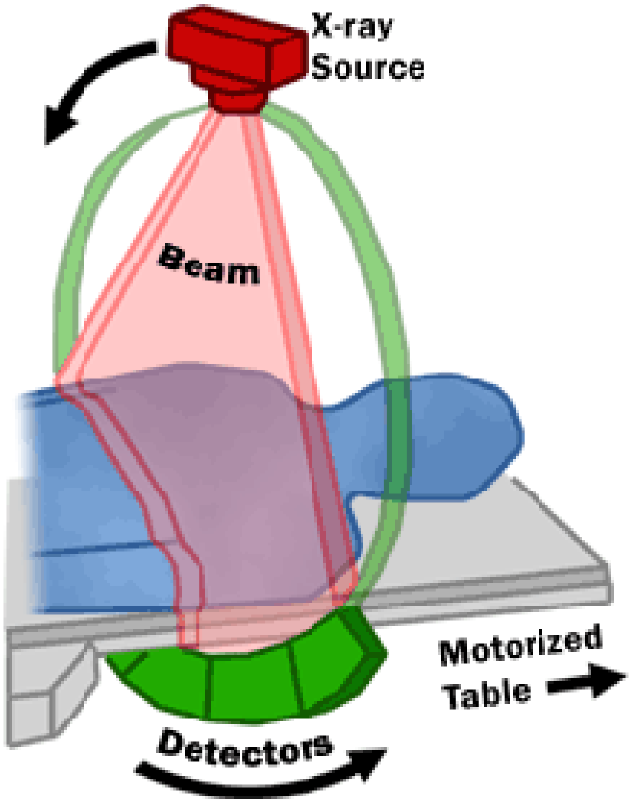
Allan M. Cormack,
born 23. Feb 1924 in
South Africa

Godfrey N. Hounsfield,
Born 28. Aug 1919 in
Newark, United Kingdom

In **computerized tomography** X-rays are produced on a ring surrounding the patient and travel through the body along many ray paths with many directions. Another ring-like detector array is recording the X-ray intensities after leaving the body. These recorded sums of absorption values of the body along the diverse X-ray paths are used to mathematically derive an X-ray image of the body.

Source: reference 2.

Usually the body is imaged ‘slice by slice’, where each slice is modelled as a 2D array of density values in the body and the X-rays are ‘summing up’ values along their trajectories.



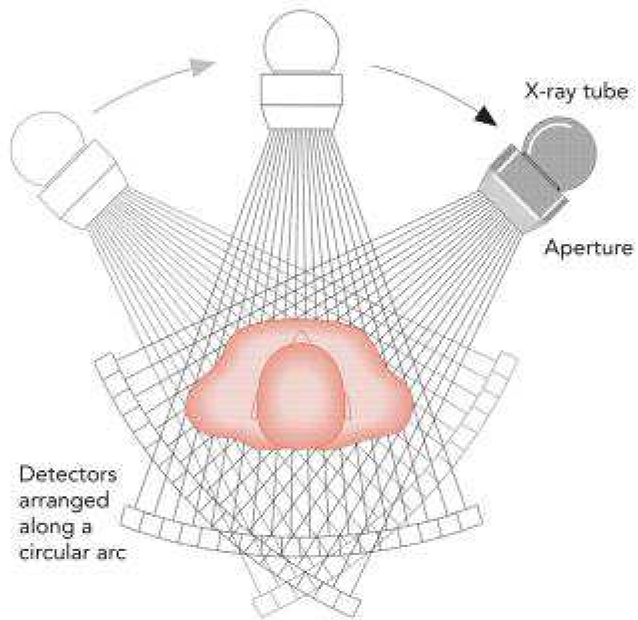
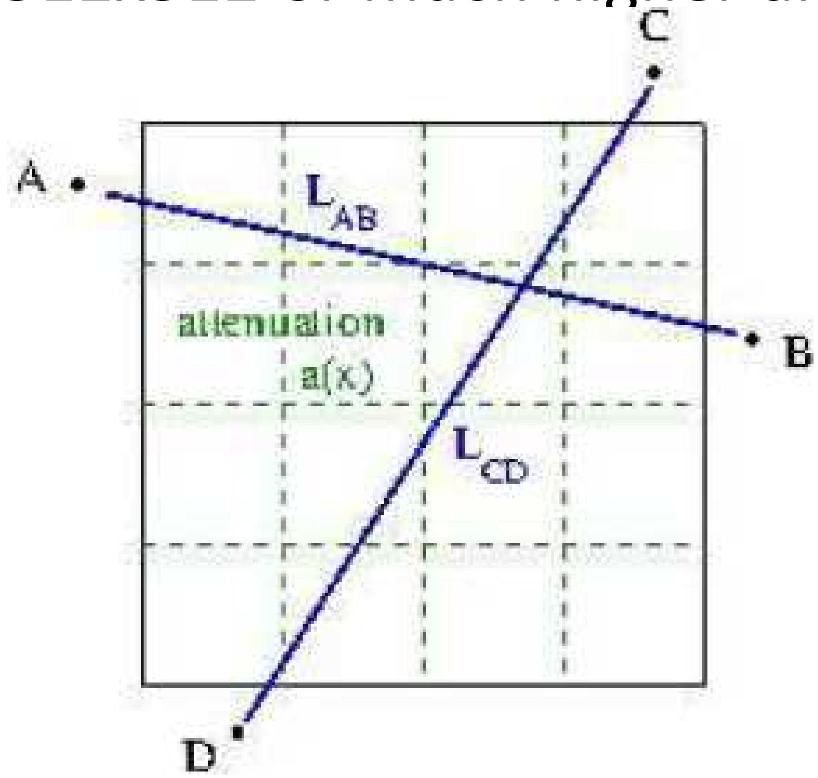


Figure 7-10 Computer tomography

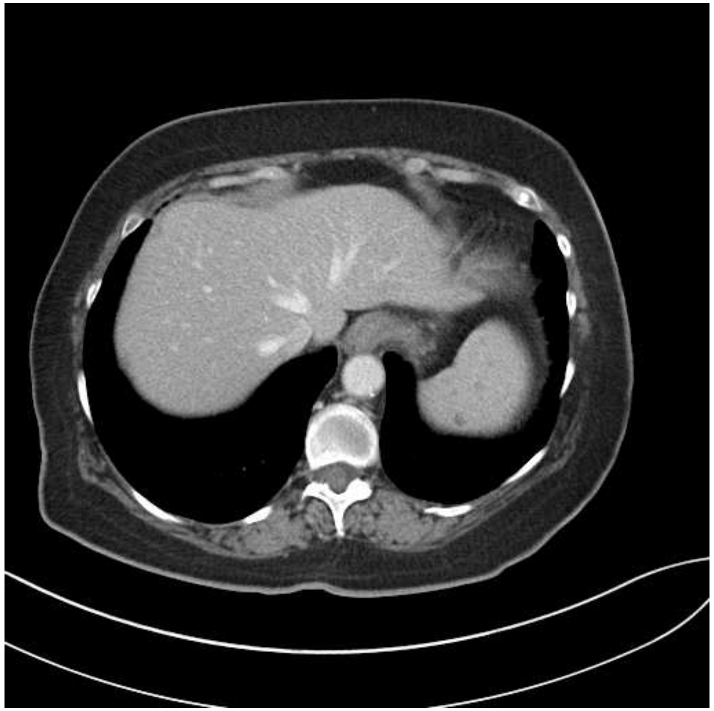
Source: references 5 and 9.

Schematic X-ray setup of one ‘slice’ of size 4x4. In practice slices of size 512x512 or much higher are used.



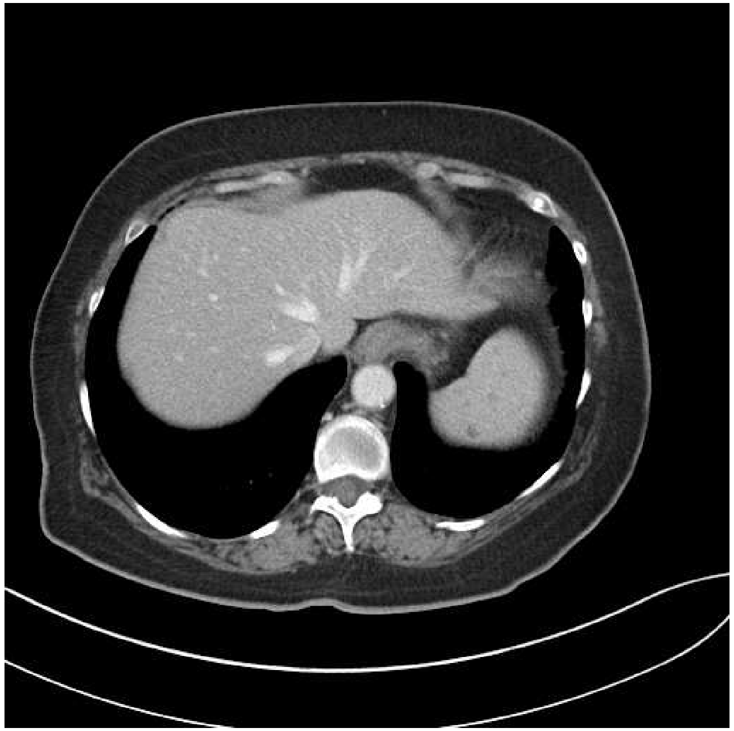
Can we identify the absorption values of the array (representing the anatomy of the human body) without ambiguities from **X-ray data**, which are just **sums over pixel values along ray paths**?

So, as in the Magic Square problem, in Computerized Tomography we want to arrange numbers in a 2D image whose sums coincides with the measured X-ray data. Each number corresponds to a grayscale value in an image as below.



Source: reference 8.

A very important question arises: can we be sure that there are not two different anatomical images which sum up to the same data along the X-ray paths, just as there were more than one Magic Square?



Source: references 8 and 5.

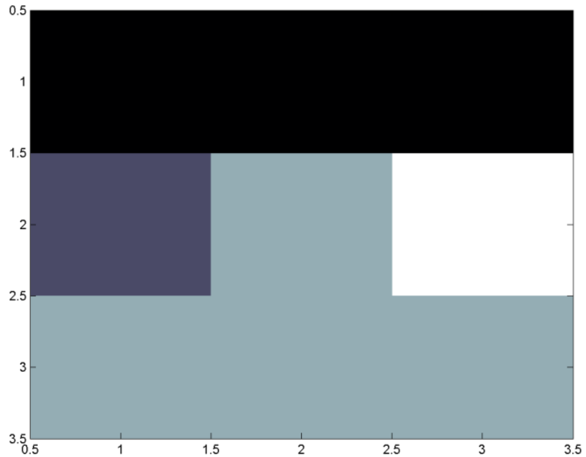
How many projections do we need in order to determine the tomographic image without any ambiguity?

Let us have a look at a simple 3x3 example. All real numbers are now allowed for filling the arrays.

The 3x3 array of numbers

$$\begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 4 & 4 & 4 \end{bmatrix}$$

The corresponding image



Remember: the brighter the pixel, the bigger the number!

First Projection: sum up along horizontal rays (i.e. rows):

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 6 \\ 3 & 4 & 5 & 12 \\ 4 & 4 & 4 & 12 \end{array}\right]$$

Second Projection: sum up along vertical rays (i.e. columns):

$$\left[\begin{array}{ccc} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 4 & 4 & 4 \\ \hline 9 & 10 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} ? & ? & ? & 6 \\ ? & ? & ? & 12 \\ ? & ? & ? & 12 \end{array}\right] \qquad \left[\begin{array}{ccc} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ \hline 9 & 10 & 11 \end{array}\right]$$

Are these two projections sufficient for unique reconstruction?

(Rule of thumb: you need ‘at least’ as many data as unknowns!)

Let us add another projection, along the main diagonal:

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & \\ 3 & 4 & 5 & 2 \\ 4 & 4 & 4 & 7 \\ \hline & 4 & 7 & 10 \end{array}\right]$$

Do we have now enough projection for determining the image?

$$\left[\begin{array}{ccc|c} ? & ? & ? & 6 \\ ? & ? & ? & 12 \\ ? & ? & ? & 12 \end{array}\right]$$

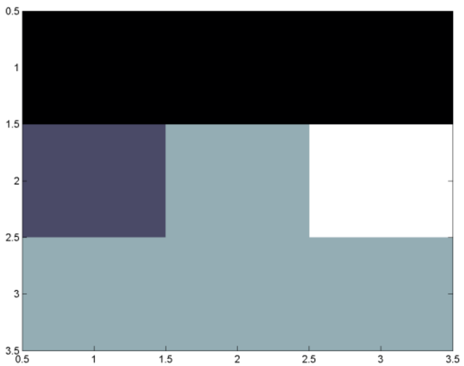
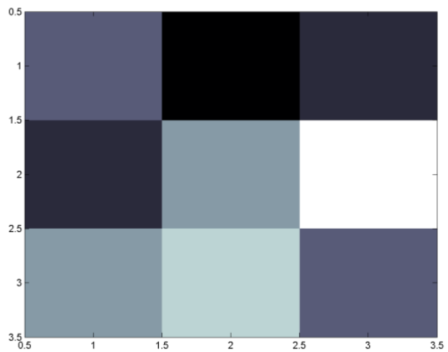
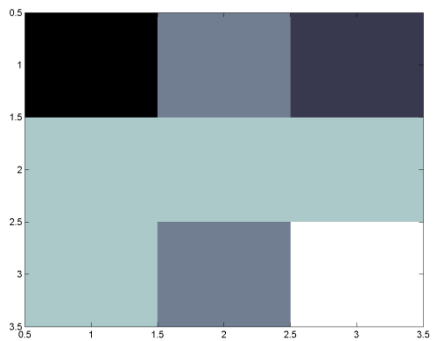
$$\left[\begin{array}{ccc} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ \hline 9 & 10 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} ? & ? & ? & \\ ? & ? & ? & 2 \\ ? & ? & ? & 7 \\ \hline & 4 & 7 & 10 \end{array}\right]$$

We have 11 data for finding 9 unknowns!

Verify that all three 'images' below satisfy all 11 data

$$\begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 4 & 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 6 \\ 4 & 5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 4 & 4 & 4 \\ 4 & 3 & 5 \end{bmatrix}$$

So, apparently the reconstruction is **still not unique**!

Remark:

In fact, following the mathematical approach of the SVD in our technical section (see below) it can be shown that all solutions of the above problem with three projections are of the form

$$\begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \\ 4 & 4 & 4 \end{bmatrix} \quad + \quad \lambda \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

for any arbitrary real number λ. It is said that

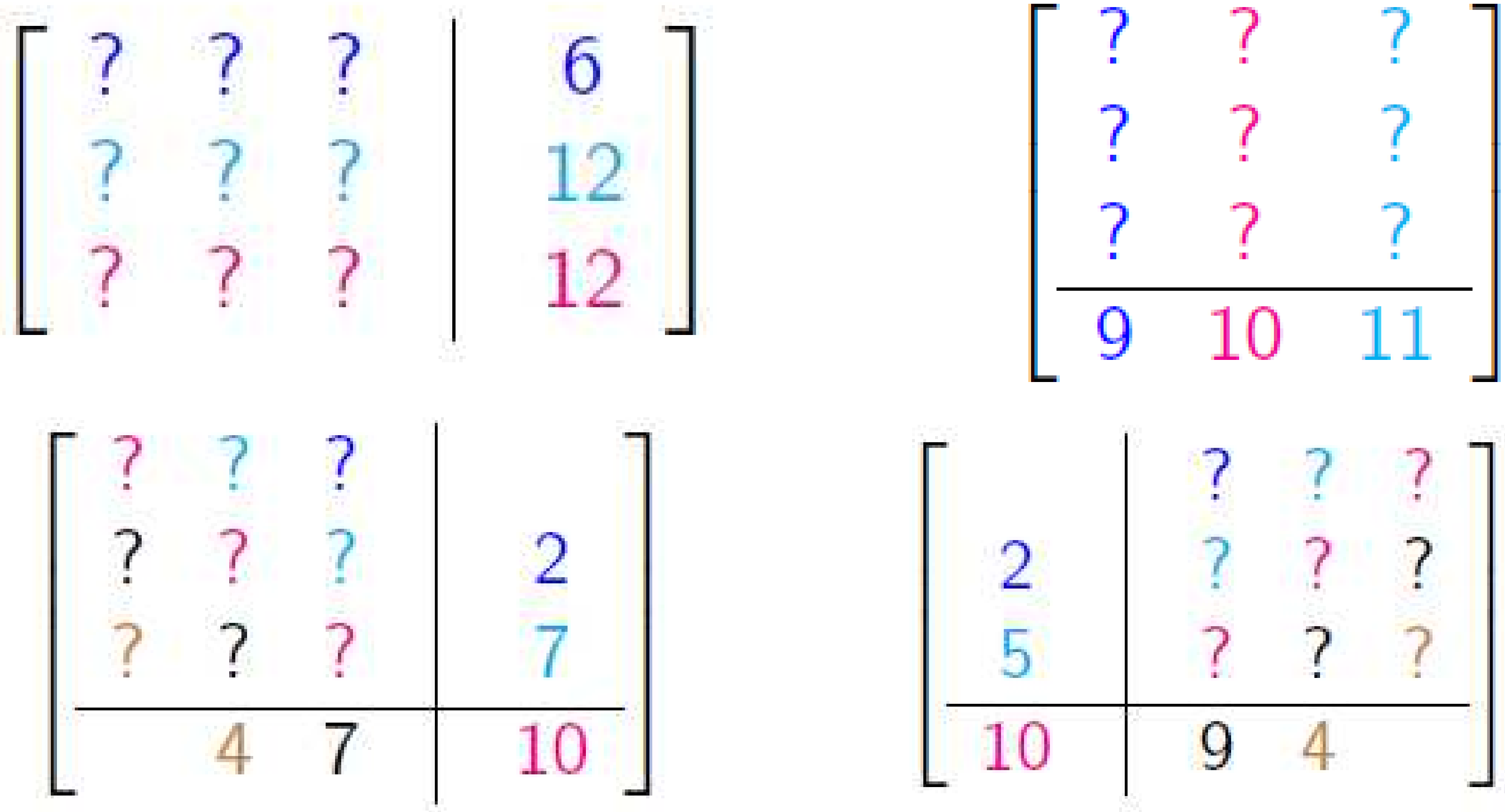
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

spans the null space of the problem.

We need to add another projection to obtain a unique result. We choose the ‘other’ diagonal direction:

	2	2	2
2	3	4	5
5	4	4	4
10	9	4	

Now we do have enough projections for determining the image without any ambiguity! (This can be proven mathematically!)



We have 16 data for finding 9 unknowns! Can you determine the 9 numbers from these 16 sums (without looking back to the true solution)? It is not that difficult now!

Okay, you figured that out. Let us look at another example. Can you find the corresponding array?

$$\left[\begin{array}{ccc|c} ? & ? & ? & 15 \\ ? & ? & ? & 15 \\ ? & ? & ? & 15 \end{array}\right] \qquad \left[\begin{array}{ccc} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ \hline 15 & 15 & 15 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} ? & ? & ? & \\ ? & ? & ? & 6 \\ ? & ? & ? & 8 \\ \hline & 4 & 12 & 15 \end{array}\right] \qquad \left[\begin{array}{c|ccc} & ? & ? & ? \\ 8 & ? & ? & ? \\ 4 & ? & ? & ? \\ \hline 15 & 16 & 2 & \end{array}\right]$$

Surprise! It is a 3x3 magic square!

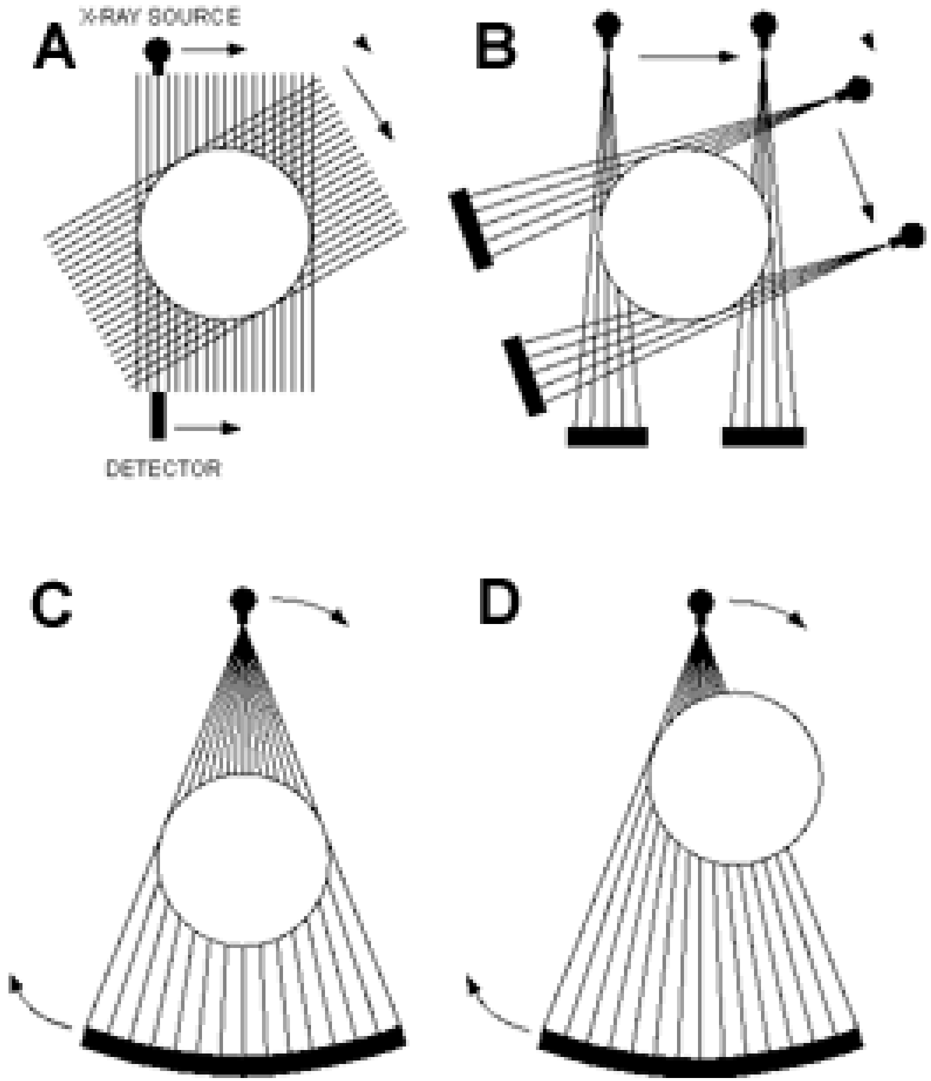
$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

By the way, the number of projections that you need for a unique result is independent of the chosen numbers in the array! For our setup, the three projections will never be sufficient for uniquely determining all numbers! Only a different geometry and arrangement of projections could reduce the number of required projections!

So, how many projections do we need in order to determine the 367x490 X-ray image values of the human spine without any ambiguity from their sums along ray-paths? (We have 367*490=179830 unknown numbers in this array!)

Source: reference 1.

Scanning Geometries



Various scanning geometries have been proposed for creating a sufficient number of independent rays for determining without ambiguity all pixel values of the CT tomogram. Amongst them:

- Parallel Beam
- Fan Beam

Source: reference 7.

In practice a trade-off needs to be found:

- Too few projections will cause non-uniqueness, which typically means artefacts in the reconstructed X-ray CT images which potentially leads to false diagnostic interpretations.
- Too many projections will increase the X-ray dose to the patient unnecessarily, which is potentially harmful to human tissue

It is a difficult mathematical problem to optimize not only the number of projections, but also their directions for specific diagnostic applications!

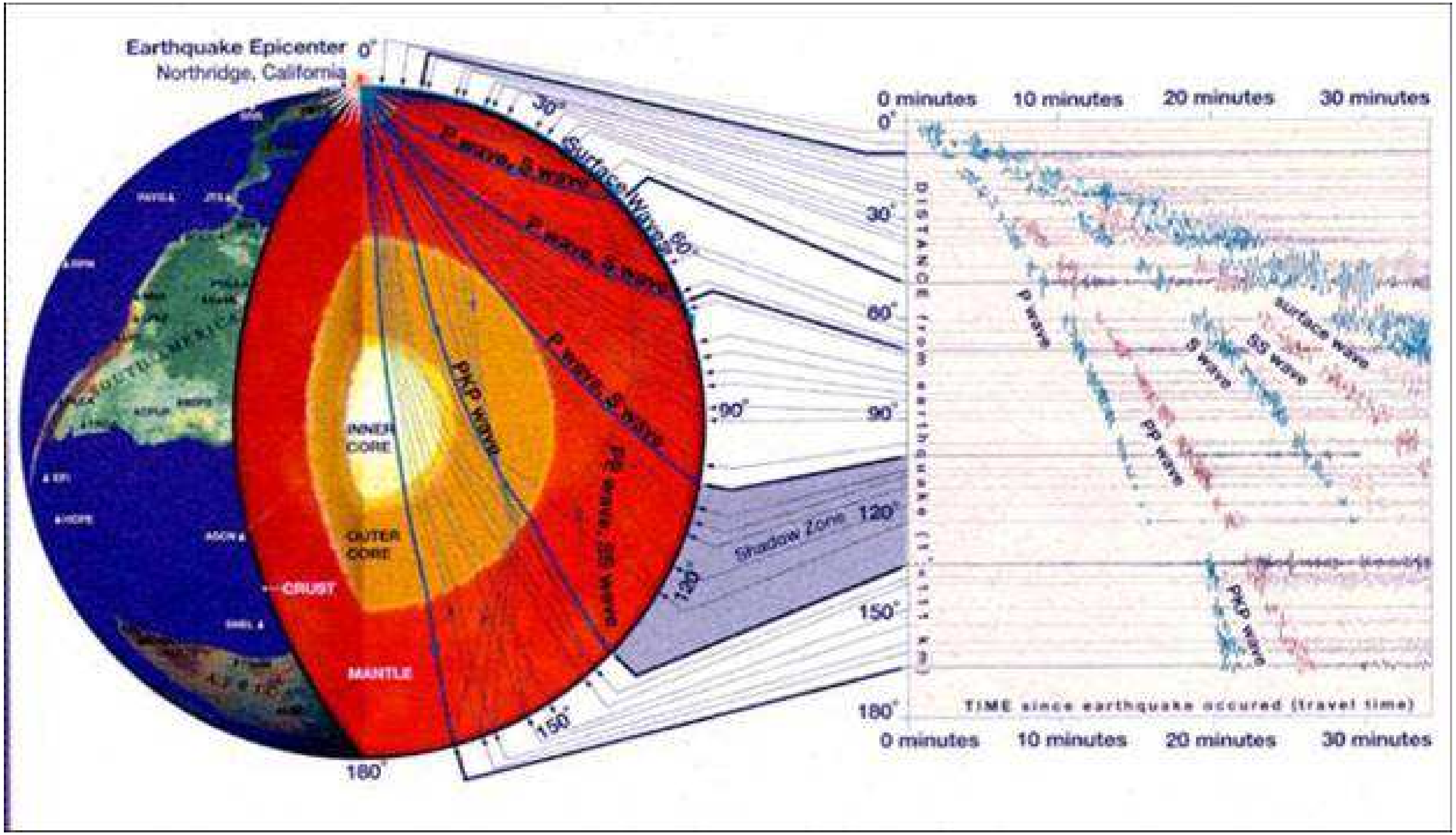
Other applications of tomography

The above ideas and concepts are not restricted to medical computerized tomography applications, or to X-ray imaging.

There exists a huge variety of imaging modalities in many important applications where similar ideas are applied.

We will present a few of them in the following.

Seismology



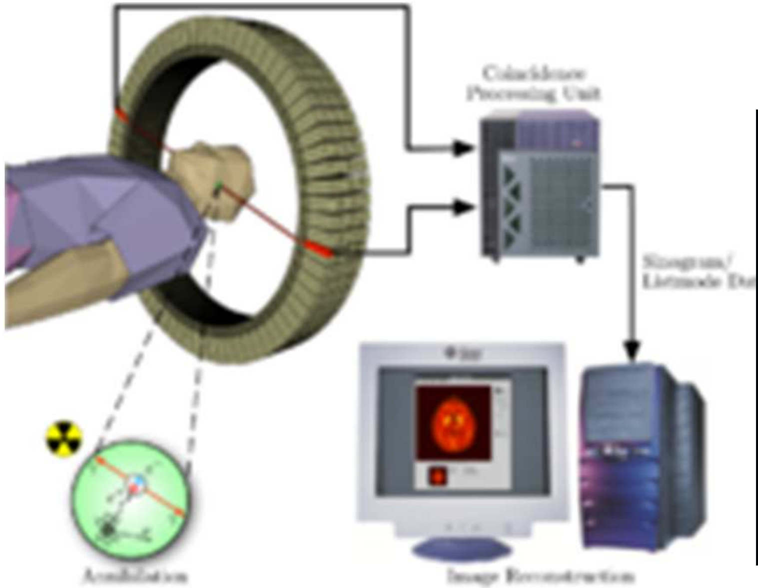
Source: reference 11.

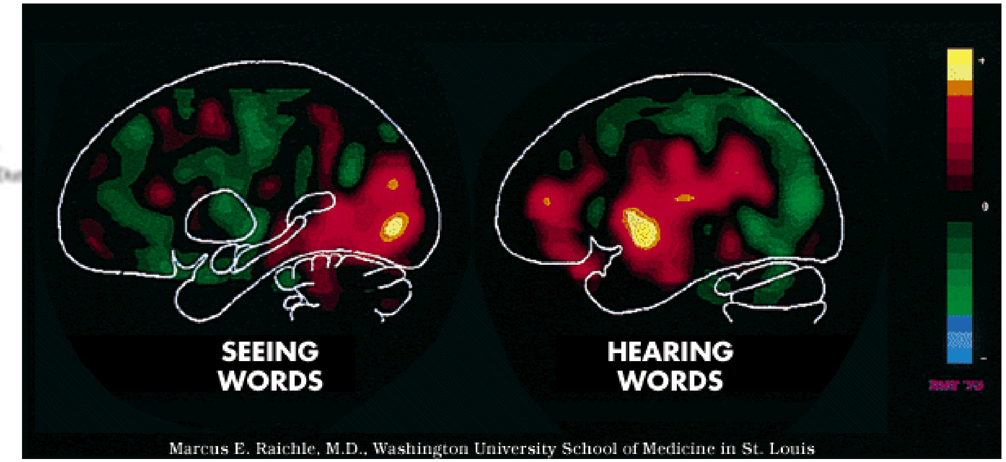
Seismology

- If an earthquake occurs at some point in the earth mantle, pressure waves created by the earthquake travel through the entire earth and can be recorded far away.
- The times of arrival of these waves are essentially sums of inverse wave velocities (called 'slownesses') along the acoustic ray paths in the earth and depend on the acoustic properties of the earth.
- From these seismograms seismologists try to obtain information about the inner structure of the Earth, e.g. the size, constitution and temperature of the inner core.

Positron Emission Tomography: in which part of the brain are we thinking at which time?

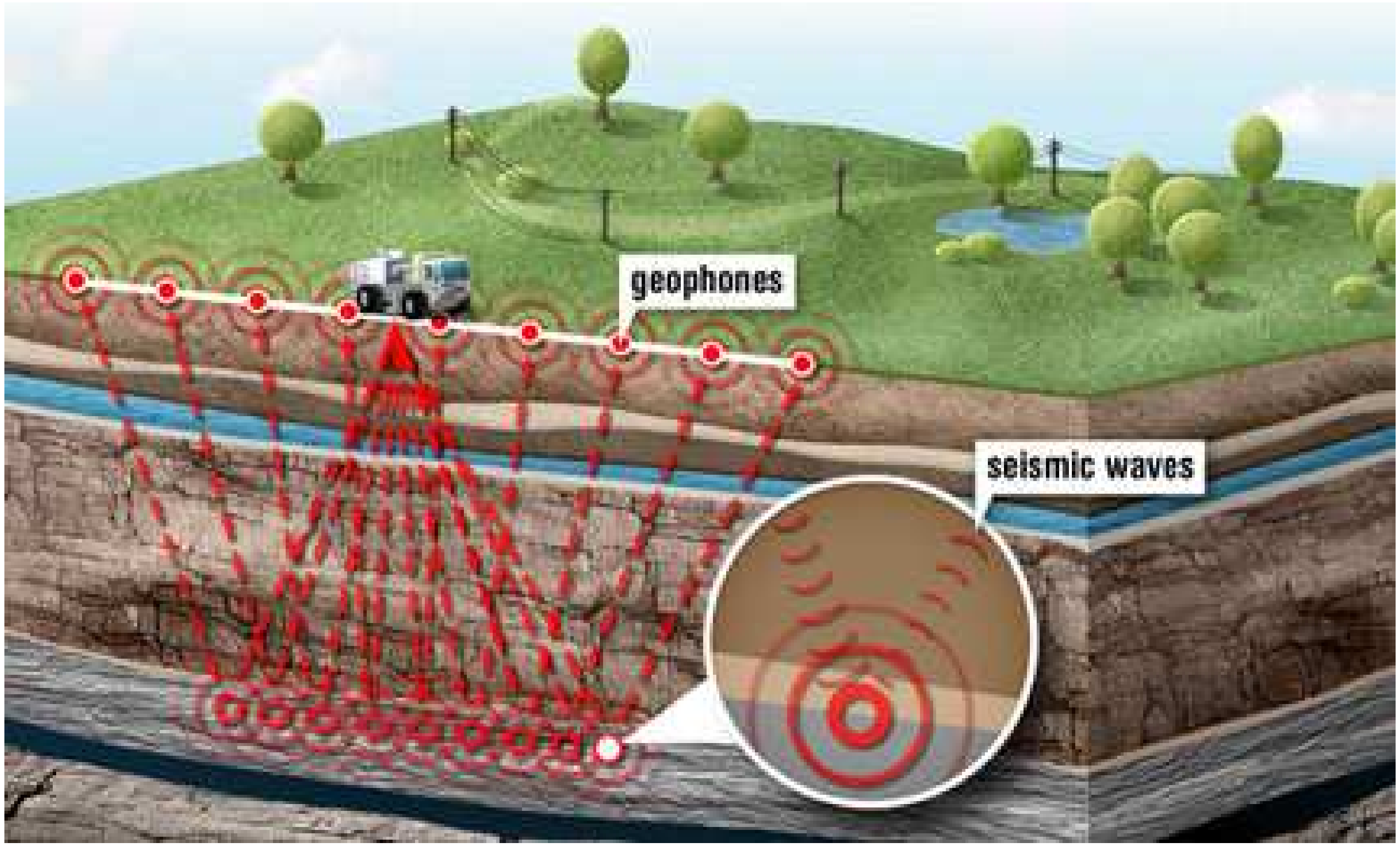
Brain activity stimulated by simple tasks can be monitored and imaged by nuclear imaging techniques.





Source: references 13 and 14.

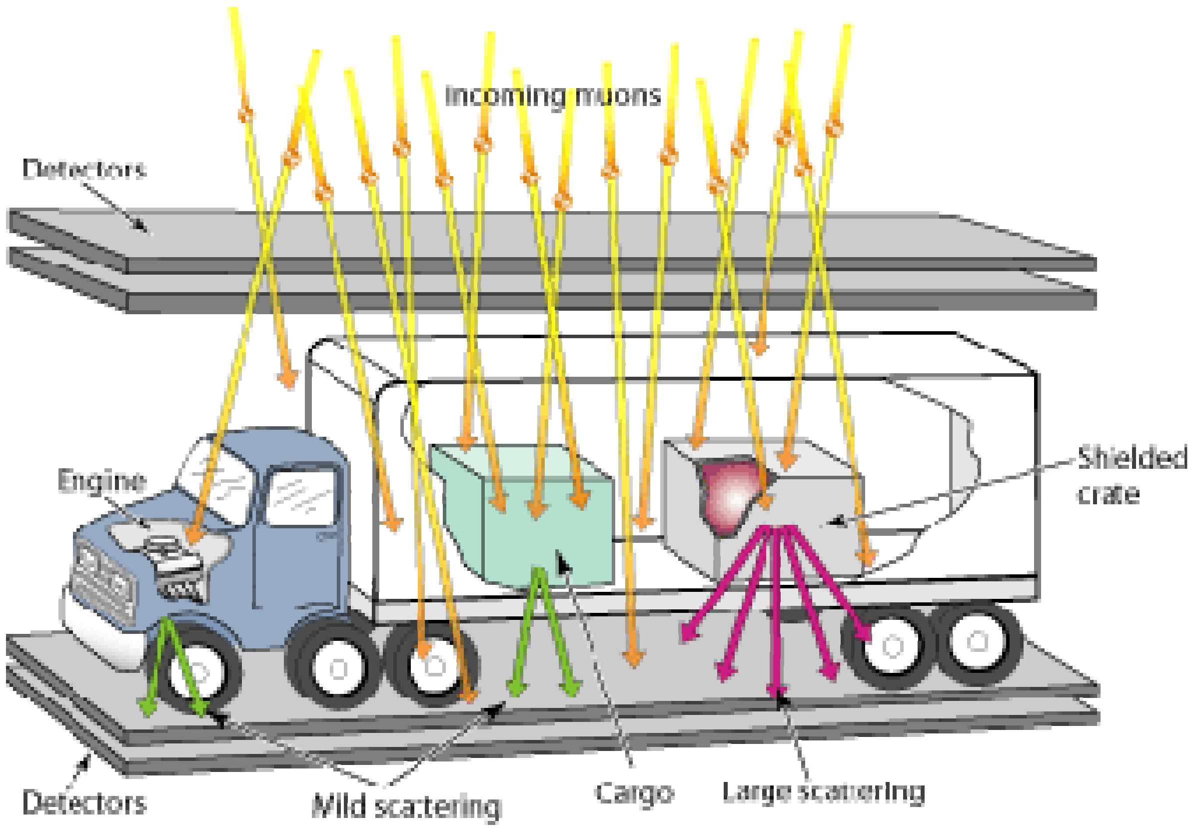
Oil and gas exploration with seismic waves



Source: reference 15.

Acoustic waves travel through the ground and are scattered back by sedimentary layers

Muon tomography at border control



Source: reference 16.

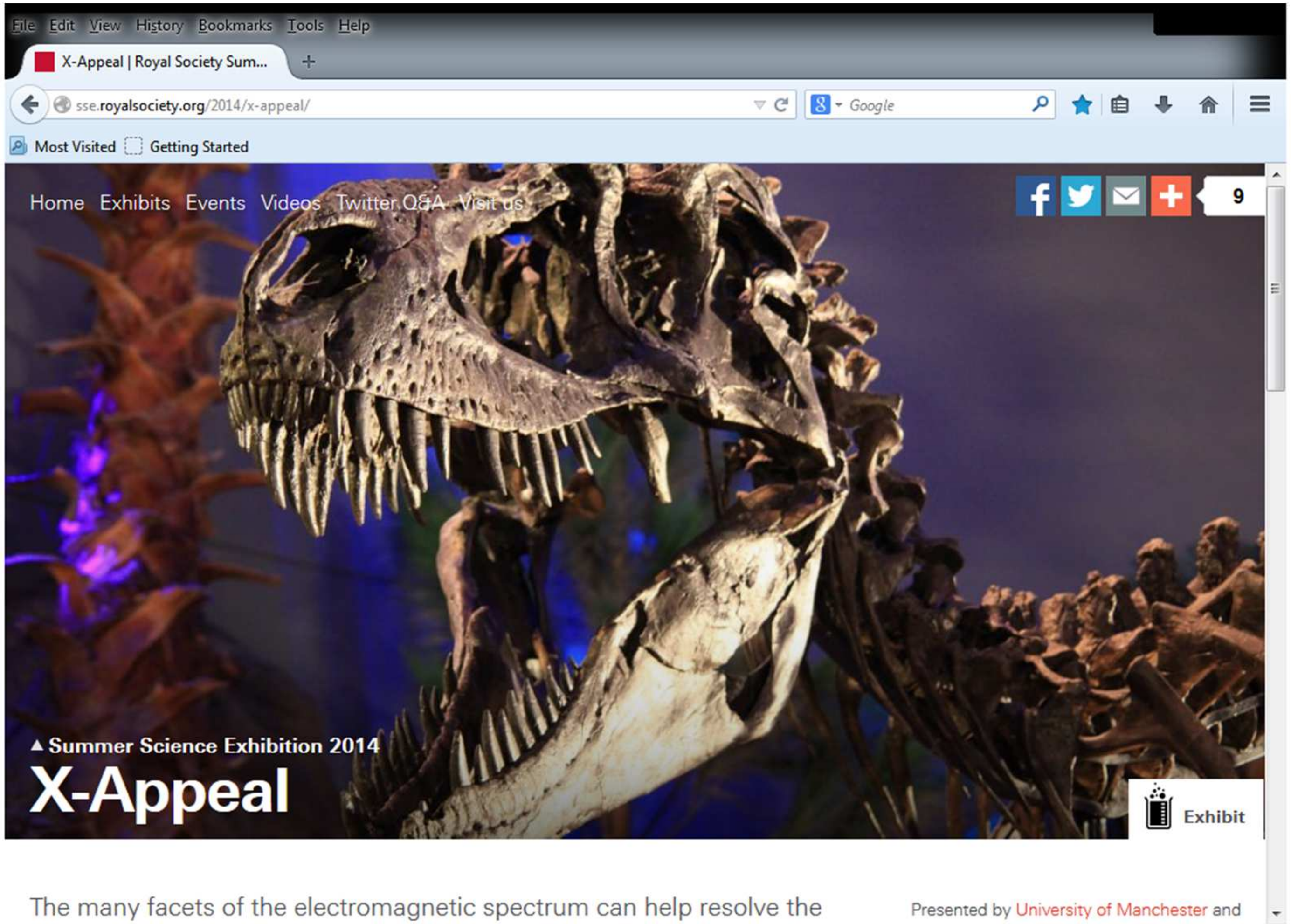
Cosmic rays ('muons') are used for detecting fissile material hidden in cargo containers.

Airport luggage scanning

Source: reference 17.

Suspicious objects need to be identified quickly and efficiently at airports before the luggage gets on the plane.

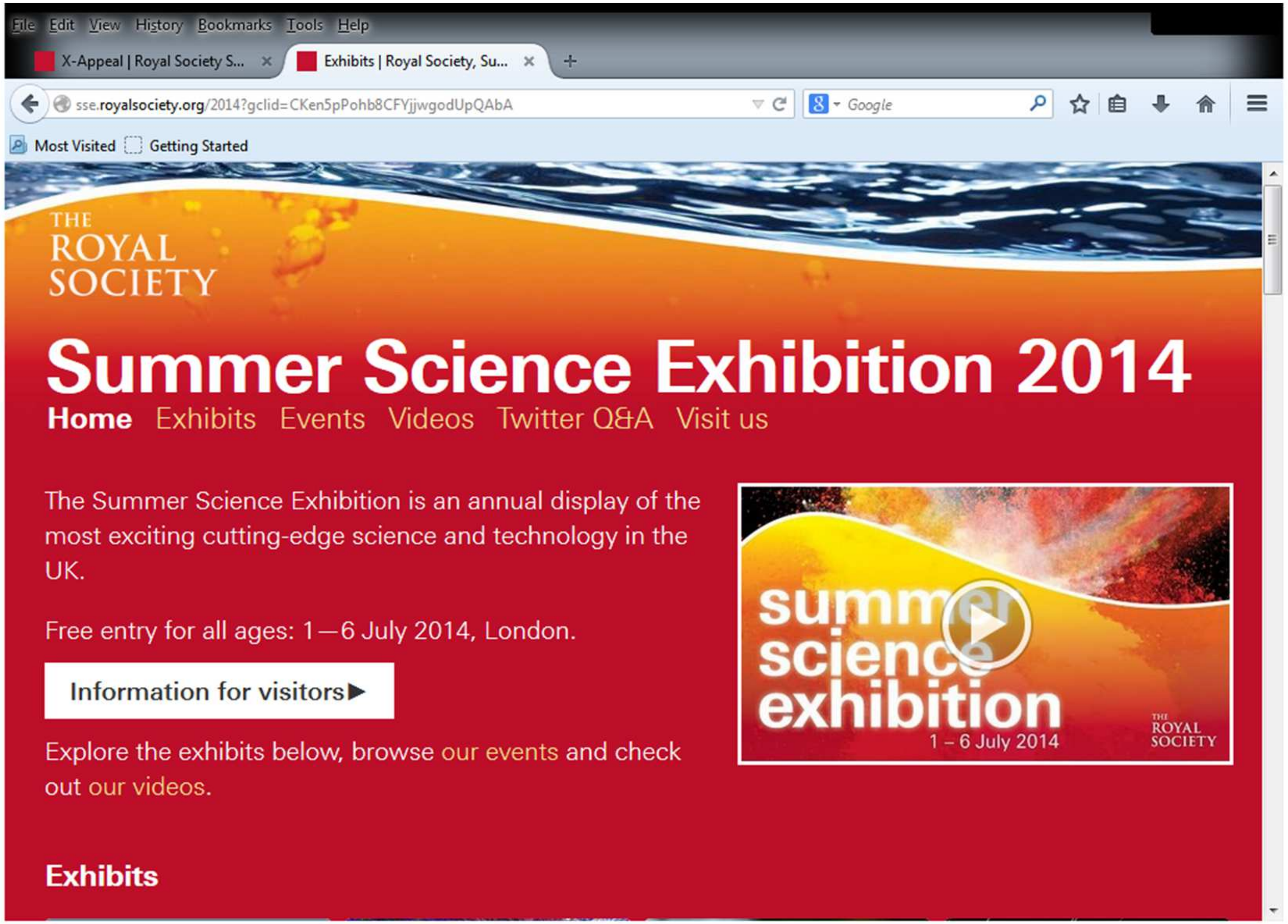
Last but not least: X-ray imaging of Dinosaurs



Source: reference 19.

Watch the movie presented at the Summer Science Exhibition 2014 in London (see the link in reference 19)!

X-Appeal is part of the Summer Science Exhibition 2014, 1-6 July 2014 in London:



Source: reference 18.

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